Chapter 17
Models in Two Space–Time Dimensions Without Interactions

17.1 Two Dimensional Model of Massless Bosons

An important motivation for our step from real numbers to integers is that we require deterministic theories to be infinitely precise. Any system based on a classical action, requires real numbers for its basic variables, but this also introduces limited precision. If, as one might be inclined to suspect, the ultimate physical degrees of freedom merely form bits and bytes, then these can only be discrete, and the prototypes of discrete systems are the integers. Perhaps later one might want to replace these by integers with a maximal size, such as integers *modulo* a prime number p, the elements of $\mathbb{Z}/p\mathbb{Z}$.

The question is how to phrase a systematic approach. For instance, how do we mimic a quantum field theory? If such a theory is based on perturbative expansions, can we mimic such expansions in terms of integers? Needless to observe that standard perturbation expansions seem to be impossible for discrete systems, but various special kinds of expansions can still be imagined, such as $1/N$ expansions, where N could be some characteristic of an underlying algebra.

We shall not be able to do this in this book, but we make a start at formulating systematic approaches. In this chapter, we consider a quantized field, whose field variables, of course, are operators with continua of eigenvalues in the real numbers. If we want to open the door to perturbative field theories, we first need to understand free particles. One example was treated in Sect. 15.2. These were fermions. Now, we try to introduce free bosons.

Such theories obey linear field equations, such as

$$\partial_t^2\phi(\vec{x},t)=\sum_{i=1}^{d}\partial_i^2\phi(\vec{x},t)-m^2\phi(\vec{x},t). \tag{17.1}$$

In the case of fermions, we succeeded, to some extent, to formulate the massless case in three space dimensions (Sect. 15.2, Sect. 15.2.3), but applying PQ theory to bosonic fields in more than two dimensions has not been successful. The problem is that equations such as Eq. (17.1) are difficult to apply to integers, even if we may fill the gaps between the integers with generators of displacements.

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In our search for systems where this can be done, we chose, as a compromise, massless fields in one space-like dimension only. The reason why this special case can be handled with *PQ* theory is, that the field equation, Eq. (17.1), can be reduced to first order equations by distinguishing left-movers and right-movers. Let us first briefly summarize the continuum quantum field theory for this case.

17.1.1 Second-Quantized Massless Bosons in Two Dimensions

We consider a single, scalar, non interacting, massless field $q(x,t)$. Both x and t are one-dimensional. The Lagrangian and Hamiltonian are:

$$\mathcal{L} = \tfrac{1}{2}\big(\partial_t q^2 - \partial_x q^2\big); \qquad H_{\mathrm{op}} = \int \mathrm{d}x\big(\tfrac{1}{2}p^2 + \tfrac{1}{2}\partial_x q^2\big), \tag{17.2}$$

where we use the symbol $p(x)$ to denote the canonical momentum field associated to the scalar field $q(x)$, which, in the absence of interactions, obeys $p(x) = \partial_t q(x)$. The fields $q(x)$ and $p(x)$ are operator fields. The equal-time commutation rules are, as usual:

$$\big[q(x), q(y)\big] = \big[p(x), p(y)\big] = 0; \qquad \big[q(x), p(y)\big] = i\delta(x-y). \tag{17.3}$$

Let us regard the time variable in $q(x,t)$ and $p(x,t)$ to be in Heisenberg notation. We then have the field equations:

$$\partial_t^2 q = \partial_x^2 q, \tag{17.4}$$

and the solution of the field equations can be written as follows:

$$a^L(x,t) = p(x,t) + \partial_x q(x,t) = a^L(x+t); \tag{17.5}$$

$$a^R(x,t) = p(x,t) - \partial_x q(x,t) = a^R(x-t). \tag{17.6}$$

The equations force the operators a^L to move to the left and a^R to move to the right. In terms of these variables, the Hamiltonian (at a given time t) is

$$H_{\mathrm{op}} = \int \mathrm{d}x \tfrac{1}{4}\Big(\big(a^L(x)\big)^2 + \big(a^R(x)\big)^2\Big). \tag{17.7}$$

The commutation rules for $a^{L,R}$ are:

$$\begin{aligned} \big[a^L, a^R\big] = 0, \quad &\big[a^L(x_1), a^L(x_2)\big] = 2i\delta'(x_1 - x_2), \\ &\big[a^R(x_1), a^R(x_2)\big] = -2i\delta'(x_1 - x_2), \end{aligned} \tag{17.8}$$

where $\delta'(z) = \frac{\partial}{\partial z}\delta(z)$.

Now let us Fourier transform in the space direction, by moving to momentum space variables k, and subtract the vacuum energy. We have in momentum space (note that we restrict ourselves to *positive* values of k):

$$a^{L,R}(k) = \tfrac{1}{\sqrt{2\pi}} \int \mathrm{d}x e^{-ikx} a^{L,R}(x), \qquad a^\dagger(k) = a(-k); \tag{17.9}$$

$$H_{\mathrm{op}} = \int_0^\infty \mathrm{d}k \, \tfrac{1}{2}\big(a^{L\dagger}(-k)a^L(-k) + a^{R\dagger}(k)a^R(k)\big), \tag{17.10}$$

$$k, k' > 0: \quad \begin{aligned} &\big[a^L(-k_1), a^L(-k_2)\big] = 0, \\ &\big[a^L(-k_1), a^{L\dagger}(-k_2)\big] = 2k_1\delta(k_1 - k_2). \end{aligned} \tag{17.11}$$

$$\begin{aligned} &\big[a^R(k_1), a^R(k_2)\big] = 0, \\ &\big[a^R(k_1), a^{R\dagger}(k_2)\big] = 2k_1\delta(k_1 - k_2). \end{aligned} \tag{17.12}$$

In this notation, $a^{L,R}(\pm k)$ are the annihilation and creation operators, apart from a factor $\sqrt{2k}$, so the Hamiltonian (17.7) can be written as

$$H_{\mathrm{op}} = \int_0^\infty \mathrm{d}k \, \big(kN^L(-k) + kN^R(k)\big). \tag{17.13}$$

where $N^{L,R}(\mp k)\mathrm{d}k$ are the occupation numbers counting the left and right moving particles. The energies of these particles are equal to the absolute values of their momentum. All of this is completely standard and can be found in all the text books about this subject.

Inserting a lattice cut-off for the UV divergences in quantum field theories is also standard practice. Restricting ourselves to integer values of the x coordinate, and using the lattice length in x space as our unit of length, we replace the commutation rules (17.3) by

$$\big[q(x), q(y)\big] = \big[p^+(x), p^+(y)\big] = 0; \qquad \big[q(x), p^+(y)\big] = i\delta_{x,y} \tag{17.14}$$

(the reason for the superscript + will be explained later, Eqs. (17.30)–(17.32)). The exact form of the Hamiltonian on the lattice depends on how we wish to deal with the lattice artefacts. The choices made below might seem somewhat artificial or special, but it can be verified that most alternative choices one can think of can be transformed to these by simple lattice field transformations, so not much generality is lost. It is important however that we wish to keep the expression (17.13) for the Hamiltonian; also on the lattice, we wish to keep the same dispersion law as in the continuum, so that all excitations must move left or right exactly with the same speed of light (which of course will be normalized to $c = 1$).

The lattice expression for the left- and right movers will be

$$a^L(x+t) = p^+(x,t) + q(x,t) - q(x-1,t); \tag{17.15}$$

$$a^R(x-t) = p^+(x,t) + q(x,t) - q(x+1,t). \tag{17.16}$$

They obey the commutation rules

$$\big[a^L, a^R\big] = 0; \qquad \big[a^L(x), a^L(y)\big] = \pm i \quad \text{if } y = x \pm 1; \quad \text{else } 0; \tag{17.17}$$

$$\big[a^R(x), a^R(y)\big] = \mp i \quad \text{if } y = x \pm 1; \quad \text{else } 0. \tag{17.18}$$

They can be seen to be the lattice form of the commutators (17.8). In momentum space, writing

$$a^{L,R}(x) \equiv \tfrac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \mathrm{d}\kappa\, a^{L,R}(\kappa) e^{i\kappa x}, \qquad a^{L,R\dagger}(\kappa) = a^{L,R}(-\kappa); \quad (17.19)$$

the commutation rules (17.11) and (17.12) are now

$$\left[a^L(-\kappa_1), a^{L\dagger}(-\kappa_2)\right] = 2\sin\kappa_1 \delta(\kappa_1 - \kappa_2); \quad (17.20)$$

$$\left[a^R(\kappa_1), a^{R\dagger}(\kappa_2)\right] = 2\sin\kappa_1 \delta(\kappa_1 - \kappa_2), \quad (17.21)$$

so our operators $a^{L,R}(\mp\kappa)$ and $a^{L,R\dagger}(\mp\kappa)$ are the usual annihilation and creation operators, multiplied by the factor

$$\sqrt{2|\sin\kappa|}. \quad (17.22)$$

If we want the Hamiltonian to take the form (17.13), then, in terms of the creation and annihilation operators (17.20) and (17.21), the Hamiltonian must be

$$H_{\mathrm{op}} = \int_0^{\pi} \mathrm{d}\kappa \frac{\kappa}{2\sin\kappa} \left(a^{L\dagger}(-\kappa)a^L(-\kappa) + a^{\dagger R}(\kappa)a^R(\kappa)\right). \quad (17.23)$$

Since, in momentum space, Eqs. (17.15) and (17.16) take the form

$$a^L(\kappa) = p^+(\kappa) + \left(1 - e^{-i\kappa}\right)q(\kappa), \qquad a^R(\kappa) = p^+(\kappa) + \left(1 - e^{i\kappa}\right)q(\kappa), \quad (17.24)$$

after some shuffling, we find the Hamiltonian (ignoring the vacuum term)

$$H_{\mathrm{op}} = \tfrac{1}{2} \int_0^{\pi} \mathrm{d}\kappa \left(\frac{\kappa}{\tan\frac{1}{2}\kappa} |p^+(\kappa)|^2 + 4k \tan\tfrac{1}{2}\kappa |q(\kappa) + \tfrac{1}{2}p^+(\kappa)|^2 \right), \quad (17.25)$$

where $|p^+(\kappa)|^2$ stands for $p^+(\kappa)p^+(-\kappa)$. Since the field redefinition $q(x) + \frac{1}{2}p^+(x) \to q(x)$ does not affect the commutation rules, and

$$\lim_{\kappa\to 0} \frac{\kappa}{2\tan(\frac{1}{2}\kappa)} = 1, \qquad 4\sin^2\left(\tfrac{1}{2}\kappa\right)\left|q(\kappa)\right|^2 \to \left|(\partial_x q)(\kappa)\right|^2, \quad (17.26)$$

we see that the continuum limit (17.2), (17.10) is obtained when the lattice length scale is sent to zero.

Because of the factor (17.22), the expression (17.23) for our Hamiltonian shows that the operators a and $a^\dagger$, annihilate and create energies of the amount $|\kappa|$, as usual, and the Hamilton equations for $a^{L,R}$ are

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} a^L(-\kappa, t) &= -i\left[a^L(-\kappa, t), H_{\mathrm{op}}\right] = \frac{-i\kappa}{2\sin\kappa} 2\sin\kappa\, a^L(-\kappa) \\ &= -i\kappa a^L(-\kappa, t); \\ \frac{\mathrm{d}}{\mathrm{d}t} a^R(\kappa, t) &= -i\kappa a^R(\kappa, t). \end{aligned} \quad (17.27)$$

Consequently,

$$\begin{aligned} a^L(-\kappa, t)e^{-i\kappa x} &= a^L(-\kappa, 0)e^{-i\kappa x - i\kappa t}; \\ a^R(\kappa, t)e^{i\kappa x} &= a^R(\kappa, 0)e^{i\kappa x - i\kappa t}. \end{aligned} \quad (17.28)$$

We now notice that the operators $a^L(x,t)=a^L(x+t)$ and $a^R(x,t)=a^R(x-t)$ move exactly one position after one unit time step. Therefore, *even on the lattice*,

$$a^L(x,1)=a^L(x+1,0),\qquad a^R(x,1)=a^R(x-1,0),\quad \text{etc.}\tag{17.29}$$

and now we can use this to eliminate $p^+(x,t)$ and $q(x,t)$ from these equations. Writing

$$p^+(x,t)\equiv p\bigl(x,t+\tfrac{1}{2}\bigr),\tag{17.30}$$

one arrives at the equations

$$q(x,t+1)=q(x,t)+p\bigl(x,t+\tfrac{1}{2}\bigr);\tag{17.31}$$

$$p\bigl(x,t+\tfrac{1}{2}\bigr)=p\bigl(x,t-\tfrac{1}{2}\bigr)+q(x-1,t)-2q(x,t)+q(x+1,t).\tag{17.32}$$

We now see why we had to shift the field $q(x,t)$ by half the field momentum in Eq. (17.25): it puts the field at the same position $t+\frac{1}{2}$ as the momentum variable $p^+(x,t)$.

Thus, we end up with a quantum field theory where not only space but also time is on a lattice. The momentum values $p(x,t+\frac{1}{2})$ can be viewed as variables on the time-like links of the lattice.

At small values of κ, the Hamiltonian (17.23), (17.25) closely approaches that of the continuum theory, and so it obeys locality conditions there. For this reason, the model would be interesting indeed, if this is what can be matched with a cellular automaton. However, there is a problem with it. At values of κ approaching $\kappa\to\pm\pi$, the kernels diverge. Suppose we would like to write the expression (17.23) in position space as

$$H_{\mathrm{op}}=\tfrac{1}{2}\sum_{x,s}M_{|s|}\bigl(a^L(x)a^L(x+s)+a^R(x)a^R(x+s)\bigr),\tag{17.33}$$

then M_s would be obtained by Fourier transforming the coefficient $\kappa/2\sin(\kappa)$ on the interval $[-\pi,\pi]$ for k. The factor $\frac{1}{2}$ comes from symmetrizing the expression (17.33) for positive and negative s. One obtains

$$M_s=\frac{1}{2\pi}\int_{-\pi+\lambda}^{\pi-\lambda}\frac{\kappa\,\mathrm{d}\kappa}{2\sin\kappa}e^{-is\kappa}=\frac{1}{2}\begin{cases}\log\frac{2}{\lambda}-\sum_{k=0}^{s/2-1}\frac{1}{k+1/2} & \text{if } s=\text{even}\\ \log(2\lambda)+\sum_{k=1}^{(s-1)/2}\frac{1}{k} & \text{if } s=\text{odd}\end{cases}\tag{17.34}$$

where λ is a tiny cut-off parameter. The divergent part of H_{op} is

$$\begin{aligned}&\tfrac{1}{4}\Bigl(\log\frac{1}{\lambda}\Bigr)\sum_{x,y}(-1)^{x-y}\bigl(a^L(x)a^L(y)+a^R(x)a^R(y)\bigr)\\&\quad=\tfrac{1}{4}\Bigl(\log\frac{1}{\lambda}\Bigr)\biggl(\Bigl(\sum_x(-1)^x a^L(x)\Bigr)^2+\Bigl(\sum_x(-1)^x a^R(x)\Bigr)^2\biggr).\end{aligned}\tag{17.35}$$

Also the kernel $4\kappa\tan\frac{1}{2}\kappa$ in Eq. (17.25) diverges as $\kappa\to\pm\pi$. Keeping the divergence would make the Hamiltonian non-local, as Eq. (17.35) shows. We can't just argue that the largest κ values require infinite energies to excite them because they do not; according to Eq. (17.13), the energies of excitations at momentum κ are merely proportional to κ itself. We therefore propose to make a smooth cut-off, replacing the divergent kernels such as $4\kappa\tan\frac{1}{2}\kappa$ by expressions such as

$$\left(4\kappa\tan\tfrac{1}{2}\kappa\right)\left(1-e^{-\Lambda^2(\pi-\kappa)^2}\right), \tag{17.36}$$

where Λ can be taken to be arbitrarily large but not infinite.

We can also say that we keep only those excitations that are orthogonal to plane waves where $a^L(x)$ or $a^R(x)$ are of the form $C(-1)^x$. Also these states we refer to as *edge states*.

What we now have is a lattice theory where the Hamiltonian takes the form (17.13), where $N^L(-\kappa)$ and $N^R(\kappa)$ (for positive κ) count excitations in the left- and the right movers, both of which move with the same speed of light at all modes. It is this system that we can now transform to a cellular automaton. Note, that even though the lattice model may look rather contrived, it has a smooth continuum limit, which would correspond to a very dense automaton, and in theories of physics, it is usually only the continuum limit that one can compare with actual observations, such as particles in field theories. We emphasize that, up this point, our system can be seen as a conventional quantum model.

17.1.2 The Cellular Automaton with Integers in 2 Dimensions

The cellular automaton that will be matched with the quantum model of the previous subsection, is a model defined on a square lattice with one space dimension x and one time coordinate t, where both x and t are restricted to be integers. The variables are two sets of integers, one set being integer numbers $Q(x,t)$ defined on the lattice sites, and the other being defined on the links connecting the point (x,t) with $(x,t+1)$. These will be called $P(x,t+\frac{1}{2})$, but they may sometimes be indicated as

$$P^+(x,t)\equiv P^-(x,t+1)\equiv P\left(x,t+\tfrac{1}{2}\right). \tag{17.37}$$

The automaton obeys the following time evolution laws:

$$Q(x,t+1)=Q(x,t)+P\left(x,t+\tfrac{1}{2}\right); \tag{17.38}$$

$$P\left(x,t+\tfrac{1}{2}\right)=P\left(x,t-\tfrac{1}{2}\right)+Q(x-1,t)-2Q(x,t)+Q(x+1,t), \tag{17.39}$$

just analogously to Eqs. (17.31) and (17.32). It is also a discrete version of a *classical* field theory where $Q(x,t)$ are the field variables and $P(x,t)=\frac{\partial}{\partial t}Q(x,t)$ are the *classical* field momenta.

Alternatively, one can write Eq. (17.39) as

$$Q(x,t+1)=Q(x-1,t)+Q(x+1,t)-Q(x,t-1), \tag{17.40}$$

which, incidentally, shows that the even lattice sites evolve independently from the odd ones. Later, this will become important.

As the reader must understand by now, Hilbert space for this system is just introduced as a tool. The basis elements of this Hilbert space are the *states* $|\{Q(x,0), P^+(x,0)\}\rangle$. If we consider *templates* as superpositions of such states, we will simply *define* the squares of the amplitudes to represent the probabilities. The total probability is the length-squared of the state, which will usually be taken to be one. At this stage, superpositions mean *nothing* more than this, and it is obvious that any chosen superposition, whose total length is one, may represent a reasonable set of probabilities. The basis elements all evolve in terms of a permutation operator that permutes the basis elements in accordance with the evolution equations (17.38) and (17.39). As a matrix in Hilbert space, this permutation operator only contains ones and zeros, and it is trivial to ascertain that statistical distributions, written as "quantum" superpositions, evolve with the same evolution matrix.

As operators in this Hilbert space, we shall introduce shift generators that are angles, defined exactly as in Eq. (16.7), but now at each point x_1 at time $t=0$, we have an operator $\kappa(x_1)$ that generates an integer shift in the variable $Q(x_1)$ and an operator $\xi^+(x_1)$ generating a shift in the integer $P^+(x_1)$:

$$e^{-i\kappa(x_1)}\big|\big\{Q,P^+\big\}\big\rangle=\big|\big\{Q'(x),P^+(x)\big\}\big\rangle;\qquad Q'(x)=Q(x)+\delta_{xx_1};\tag{17.41}$$

$$e^{i\xi^+(x_1)}\big|\big\{Q,P^+\big\}\big\rangle=\big|\big\{Q(x),P^{+\prime}(x)\big\}\big\rangle;\qquad P^{+\prime}(x)=P^+(x)+\delta_{xx_1};\tag{17.42}$$

The time variable t is an integer, so what our evolution equations (17.38) and (17.39) generate is an operator $U_{\mathrm{op}}(t)$ obeying $U_{\mathrm{op}}(t_1+t_2)=U_{\mathrm{op}}(t_1)U_{\mathrm{op}}(t_2)$, but only for integer time steps. In Sect. 12.2, Eq. (12.10), a Hamiltonian H_{op} was found that obeys $U_{\mathrm{op}}(t)=e^{-iH_{\mathrm{op}}t}$, by Fourier analysis. The problem with that Hamiltonian is that

1. It is not unique: one may add 2π times any integer to any of its eigenvalues; and
2. It is not extensive: if two parts of a system are space-like separated, we would like the Hamiltonian to be the sum of the two separate Hamiltonians, but then it will quickly take values more than π, whereas, by construction, the Hamiltonian (12.10) will obey $|H|\le\pi$.

Thus, by adding appropriate multiples of real numbers to its eigenvalues, we would like to transform our Hamiltonian into an extensive one. The question is how to do this.

Indeed, this is one of the central questions that forced us to do the investigations described in this book; the Hamiltonian of the quantum field theory considered here is an extensive one, and also naturally bounded from below.

At first sight, the similarity between the automaton described by the equations (17.38) and (17.39), and the quantum field theory of section (17.1.1) may seem to be superficial at best. Quantum physicists will insist that the quantum theory is fundamentally different.

However, we claim that there is an *exact mapping* between the basis elements of the quantized field theory of Sect. 17.1.1 and the states of the cellular automaton (again, with an exception for possible edge states). We shall show this by concentrating on the left-movers and the right-movers separately.

Our procedure will force us first to compare the left-movers and the right-movers separately in both theories. The automaton equations (17.38) and (17.39) ensure that, if we start with integers at $t = 0$ and $t = \frac{1}{2}$, all entries at later times will be integers as well. So this is a discrete automaton. We now introduce the combinations $A^L(x,t)$ and $A^R(x,t)$ as follows (all these capital letter variables take integer values only):

$$A^L(x,t) = P^+(x,t) + Q(x,t) - Q(x-1,t); \tag{17.43}$$

$$A^R(x,t) = P^+(x,t) + Q(x,t) - Q(x+1,t), \tag{17.44}$$

and we derive

$$\begin{aligned} A^L(x,t+1) &= P^+(x,t) + Q(x-1,t+1) - 2Q(x,t+1) + Q(x-1,t+1) \\ &\quad + Q(x,t+1) - Q(x-1,t+1) \\ &= P^+(x,t) + Q(x-1,t+1) - Q(x,t+1) \\ &= P^+(x,t) + Q(x-1,t) + P^+(x-1,t) - Q(x,t) - P^+(x,t) \\ &= P^+(x-1,t) + Q(x-1,t) - Q(x,t) = A^L(x-1,t). \end{aligned} \tag{17.45}$$

So, we have

$$A^L(x-1,t+1) = A^L(x,t) = A^L(x+t); \qquad A^R(x,t) = A^R(x-t), \tag{17.46}$$

which shows that A^L is a left-mover and A^R is a right mover. All this is completely analogous to Eqs. (17.15) and (17.16).

17.1.3 The Mapping Between the Boson Theory and the Automaton

The states of the quantized field theory on the lattice were generated by the left- and right moving operators $a^L(x+t)$ and $a^R(x-t)$, where x and t are integers, but a^L and a^R have continua of eigenvalues, and they obey the commutation rules (17.17) and (17.18):

$$\left[a^L, a^R\right] = 0; \qquad \left[a^L(x), a^L(y)\right] = \pm i \quad \text{if } y = x \pm 1; \quad \text{else } 0; \tag{17.47}$$

$$\left[a^R(x), a^R(y)\right] = \mp i \quad \text{if } y = x \pm 1; \quad \text{else } 0. \tag{17.48}$$

In contrast, the automaton has integer variables $A^L(x+t)$ and $A^R(x-t)$, Eqs, (17.43) and (17.44). They live on the same space–time lattice, but they are integers, and they commute.

Now, PQ theory suggests what we have to do. The shift generators $\kappa(x_1)$ and $\xi(x_1)$ (Eqs. (17.41) and (17.42)) can be combined to define shift operators $\eta^L(x_1)$ and $\eta^R(x_1)$ for the integers $A^L(x_1,t)$ and $A^R(x_1,t)$. Define

$$e^{i\eta^L(x_1)}\left|\left\{A^L, A^R\right\}\right\rangle = \left|\left\{A^{L'}, A^R\right\}\right\rangle, \qquad A^{L'}(x) = A^L(x) + \delta_{x,x_1}, \tag{17.49}$$

$$e^{i\eta^R(x_1)}\left|\left\{A^L, A^R\right\}\right\rangle = \left|\left\{A^L, A^{R'}\right\}\right\rangle, \qquad A^{R'}(x) = A^R(x) + \delta_{x,x_1}. \tag{17.50}$$

These then have to obey the following equations:

$$\begin{aligned}\xi(x) &= \eta^L(x) + \eta^R(x);\\ -\kappa(x) &= \eta^L(x) + \eta^R(x) - \eta^L(x+1) - \eta^R(x-1).\end{aligned} \tag{17.51}$$

The first of these tells us that, according to Eqs. (17.43) and (17.44), raising $P^+(x)$ by one unit, while keeping all others fixed, implies raising this combination of A^L and A^R. The second tells us what the effect is of raising only $Q(x)$ by one unit while keeping the others fixed. Of course, the additions and subtractions in Eqs. (17.51) are *modulo* 2π.

Inverting Eqs. (17.51) leads to

$$\begin{aligned}\eta^L(x+1) - \eta^L(x-1) &= \xi(x) + \kappa(x) - \xi(x-1),\\ \eta^R(x-1) - \eta^R(x+1) &= \xi(x) + \kappa(x) - \xi(x+1).\end{aligned} \tag{17.52}$$

These are difference equations whose solutions involve infinite sums with a boundary assumption. This has no further consequences; we take the theory to be defined by the operators $\eta^{L,R}(x)$, not the $\xi(x)$ and $\kappa(x)$. As we have encountered many times before, there are some non-local modes of measure zero, $\eta^L(x+2n) =$ constant, and $\eta^R(x+2n) =$ constant.

What we have learned from the PQ theory, is that, in a sector of Hilbert space that is orthogonal to the edge state, an integer variable A, and its shift operator η, obey the commutation rules

$$Ae^{i\eta} = e^{i\eta}(A+1); \qquad [\eta, A] = i. \tag{17.53}$$

This gives us the possibility to generate operators that obey the commutation rules (17.47) and (17.48) of the quantum field theory:

$$a^L(x) \stackrel{?}{=} \sqrt{2\pi}A^L(x) - \frac{1}{\sqrt{2\pi}}\eta^L(x-1); \tag{17.54}$$

$$a^R(x) \stackrel{?}{=} \sqrt{2\pi}A^R(x) - \frac{1}{\sqrt{2\pi}}\eta^R(x+1). \tag{17.55}$$

The factors $\sqrt{2\pi}$ are essential here. They ensure that the spectrum is not larger or smaller than the real line, that is, without gaps or overlaps (degeneracies).

The procedure can be improved. In the expressions (17.54) and (17.55), we have an edge state whenever $\eta^{L,R}$ take on the values $\pm\pi$. This is an unwanted situation: these edge states make all wave functions discontinuous on the points $a^{L,R}(x) = \sqrt{2\pi}(N(x) + \frac{1}{2})$. Fortunately, we can cancel most of these edge states by repeating more precisely the procedure explained in our treatment of PQ theory: these states were due to vortices in two dimensional planes of the tori spanned by the η variables. Let us transform, by means of standard Fourier transforming the A lattices to the η circles, so that we get a multi-dimensional space of circles—one circle at every point x.

As in the simple PQ theory (see Eqs. (16.14) and (16.15)), we can introduce a phase function $\varphi(\{\eta^L\})$ and a $\varphi(\{\eta^R\})$, so that Eqs. (17.54) and (17.55) can be replaced with

$$a^L(x) = -i\sqrt{2\pi}\frac{\partial}{\partial\eta^L(x)} + \sqrt{2\pi}\left(\frac{\partial}{\partial\eta^L(x)}\varphi\big(\{\eta^L\}\big)\right) - \frac{1}{\sqrt{2\pi}}\eta^L(x-1), \quad (17.56)$$

$$a^R(x) = -i\sqrt{2\pi}\frac{\partial}{\partial\eta^R(x)} + \sqrt{2\pi}\left(\frac{\partial}{\partial\eta^R(x)}\varphi\big(\{\eta^R\}\big)\right) - \frac{1}{\sqrt{2\pi}}\eta^R(x+1), \quad (17.57)$$

where $\varphi(\{\eta(x)\})$ is a phase function with the properties

$$\varphi\big(\big\{\eta^L(x) + 2\pi\delta_{x,x_1}\big\} = \varphi\big(\big\{\eta^L(x)\big\}\big) + \eta^L(x_1+1); \quad (17.58)$$

$$\varphi\big(\big\{\eta^R(x) + 2\pi\delta_{x,x_1}\big\} = \varphi\big(\big\{\eta^R(x)\big\}\big) + \eta^R(x_1-1). \quad (17.59)$$

Now, as one can easily check, the operators $a^{L,R}(x)$ are exactly periodic for all η variables, just as we had in Sect. 16.

A phase function with exactly these properties can be written down. We start with the elementary function $\phi(\kappa,\xi)$ derived in Sect. 16.2, Eq. (16.28), having the properties

$$\phi(\kappa,\xi+2\pi) = \phi(\kappa,\xi)+\kappa; \qquad \phi(\kappa+2\pi,\xi) = \phi(\kappa,\xi); \quad (17.60)$$

$$\phi(\kappa,\xi) = -\phi(-\kappa,\xi) = -\phi(\kappa,-\xi); \qquad \phi(\kappa,\xi)+\phi(\xi,\kappa) = \kappa\xi/2\pi. \quad (17.61)$$

The function obeying Eqs. (17.58) and (17.59) is now not difficult to construct:

$$\begin{aligned} \varphi\big(\{\eta^L\}\big) &= \sum_x \phi\big(\eta^L(x+1), \eta^L(x)\big); \\ \varphi\big(\{\eta^R\}\big) &= \sum_x \phi\big(\eta^R(x-1), \eta^R(x)\big), \end{aligned} \quad (17.62)$$

and as was derived in Sect. 16.2, a phase function with these properties can be given as the phase of an elliptic function,

$$r(\kappa,\xi)e^{i\phi(\kappa,\xi)} \equiv \sum_{N=-\infty}^{\infty} e^{-\pi(N-\frac{\xi}{2\pi})^2 - iN\kappa}, \quad (17.63)$$

where r and ϕ are both real functions of κ and ξ.

We still have edge states, but now these only sit at the corners where two consecutive η variables take the values $\pm\pi$. This is where the phase function ϕ, and therefore also φ, become singular. We suspect that we can simply ignore them.

We then reach an important conclusion. The states of the cellular automaton can be used as a basis for the description of the quantum field theory. These models are equivalent. This is an astounding result. For generations we have been told by our physics teachers, and we explained to our students, that quantum theories are fundamentally different from classical theories. No-one should dare to compare a simple computer model such as a cellular automaton based on the integers, with a fully quantized field theory. Yet here we find a quantum field system and an automaton that are based on states that neatly correspond to each other, they evolve identically. If we describe some probabilistic distribution of possible automaton states using Hilbert space as a mathematical device, we can use *any* wave function, certainly also waves in which the particles are *entangled*, and yet these states evolve exactly the same way.

Physically, using 19th century logic, this should have been easy to understand: when quantizing a classical field theory, we get energy packets that are quantized and behave as particles, but exactly the same are generated in a cellular automaton based on the integers; these behave as particles as well. Why shouldn't there be a mapping?

Of course one can, and should, be skeptical. Our field theory was not only constructed without interactions and without masses, but also the wave function was devised in such a way that it cannot spread, so it should not come as a surprise that no problems are encountered with interference effects, so yes, all we have is a primitive model, not very representative for the real world. Or is this just a beginning?

Note: being exactly integrable, the model discussed in this section has infinitely many conservation laws. For instance, one may remark that the equation of motion (17.40) does not mix even sites with odd sites of the lattice; similar equations select out sub-lattices with $x + t = 4k$ and x and t even, from other sub-lattices.

17.1.4 An Alternative Ontological Basis: The Compactified Model

In the above chapters and sections of this chapter, we have seen various examples of deterministic models that can be mapped onto quantum models and back. The reader may have noticed that, in many cases, these mappings are not unique. Modifying the choices of the constant energy shifts δE_i in the composite cogwheel model, Sect. 2.2.2, we saw that many apparently different quantum theories can be mapped onto the same set of cogwheels, although there, the δE_i could have been regarded as various chemical potentials, having no effect on the evolution law. In our PQ theory, one is free to add fractional constants to Q and P, thus modifying the mapping somewhat. Here, the effect of this would be that the ontological states obtained from one mapping do not quite coincide with those of the other, they are superpositions, and this is an example of the occurrence of sets of ontological states that are not equivalent, but all equally legal.

The emergence of inequivalent choices of an ontological basis is particularly evident if the quantum system in question has symmetry groups that are larger than those of the ontological system. If the ontological system is based on a lattice, it can only have some of the discrete lattice groups as its symmetries, whereas the quantum system, based on real coordinates, can have continuous symmetry groups such as the rotation, translation and Lorentz group. Performing a symmetry transformation that is not a symmetry of the ontological model then leads to a new set of ontological states (or "wave functions") that are superpositions of the other states. Only one of these sets will be the "real" ontological states. For our theory, and in particular the cellular automaton interpretation, Chaps. 5 and 21, this has no further consequences, except that it will be almost impossible to single out the "true" ontological basis as opposed to the apparent ones, obtained after quantum symmetry transformations.

In this subsection, we point out that even more can happen. Two (or perhaps more) systems of ontological basis elements may exist that are entirely different from one another. This is the case for the model of the previous subsection, which handles the mathematics of non-interacting massless bosons in $1+1$ dimensions. We argued that an ontological basis is spanned by all states where the field $q(x,t)$ is replaced by integers $Q(x,t)$. A lattice in x,t was introduced, but this was a temporary lattice; we could send the mesh size to zero in the end.

In Sect. 17.1.2, we introduced the integers $A^L(x)$ and $A^R(x)$, which are the integer-valued left movers and right movers; they span an ontological basis. Equivalently, one could have taken the integers $Q(x,t)$ and $P^+(x,t)$ at a given time t, but this is just a reformulation of the same ontological system.

But why not take the continuous degrees of freedom $\eta^L(x)$ and $\eta^R(x)$ (or equivalently, $\xi(x,t)$ and $\kappa(x,t)$)? At each x, these variables take values between $-\pi$ and π. Since they are also left- and right movers, their evolution law is exactly as deterministic as that of the integers A^L and A^R:

$$\frac{\partial}{\partial t}\eta^L(x,t)=\frac{\partial}{\partial x}\eta^L(x,t),\qquad \frac{\partial}{\partial t}\eta^R(x,t)=-\frac{\partial}{\partial x}\eta^R(x,t),\tag{17.64}$$

while all η's commute.

Actually, for the η fields, it is much easier now to regard the continuum limit for the space–time lattice. The *quantum operators* $a^{L,R}$ are still given by Eqs, (17.56) and (17.57). There is a singularity when two consecutive η fields take the values $\pm\pi$, but if they don't take such values at $t=t_0$, they never reach those points at other times.

There exists a somewhat superior way to rephrase the mapping by making use of the fact that the η fields are continuous, so that we can do away with, or hide, the lattice. This is shown in more detail in Sect. 17.3.5, where these ideas are applied in string theory.

What we conclude from this subsection is that *both* our quantum model of bosons *and* the model of left and right moving integers are mathematically equivalent to a classical theory of scalar fields where the values are only defined *modulo* 2π. From the ontological point of view, this new model is entirely different from both previous models.

Because the variables of the classical model only take values on the circle, we call the classical model a *compactified* classical field theory. At other places in this book, the author warned that classical, continuous theories may not be the best ontological systems to assume for describing Nature, because they tend to be *chaotic*: as time continues, more and more decimal places of the real numbers describing the initial state will become relevant, and this appears to involve unbounded sets of digital data. To our present continuous field theory in $1+1$ dimensions, this objection does not apply, because there is no chaos; the theory is entirely integrable. Of course, in more complete models of the real world we do not expect integrability, so there this objection against continuum models does apply.

17.1.5 The Quantum Ground State

Nevertheless, the mappings we found are delicate ones, and not always easy to implement. For instance, one would like to ask which solution of the cellular automaton, or the compactified field theory, corresponds to the quantum ground state of the quantized field theory. First, we answer the question: if you have the ground state, how do we add a single particle to it?

Now this should be easy. We have the creation and annihilation operators for left movers and right movers, which are the Fourier transforms of the operators $a^{L,R}(x)$ of Eqs. (17.56) and (17.57). When the Fourier parameter, the lattice momentum κ, is in the interval $-\pi < \kappa < 0$, then $a^L(\kappa)$ is an annihilation operator and $a^R(\kappa)$ is a creation operator. When $0 < \kappa < \pi$, this is the other way around. Since nothing can be annihilated from the vacuum state, the annihilation operators vanish, so $a^{L,R}(x)$ acting on the vacuum can only give a superposition of one-particle states.

To see how a single left-moving particle is added to a classical cellular automaton state, we consider the expression (17.56) for $a^L(x)$, acting on the left-mover's coordinate $x+t \to x$ when $t=0$. The operator $\partial/\partial\eta^L(x)$ multiplies the amplitude for the state with $iA^L(x)$; the other operators in Eq. (17.56) are just functions of η^L at the point x and the point $x-1$. Fourier transforming these functions gives us the operators $e^{\pm Ni\eta^L}$ multiplied with the Fourier coefficients found, acting on our original state. According to Eq. (17.49), this means that, at the two locations x and $x-1$, we add or subtract N units to the number A^L there, and then we multiply the new state with the appropriate Fourier coefficient. Since the functions in question are bounded, we expect the Fourier expansion to converge reasonably well, so we can regard the above as being a reasonable answer to the question how to add a particle. Of course, our explicit construction added a particle at the point x. Fourier transforming it, gives us a particle with momentum $-\kappa$ and energy κ.

In the compactified field model, the action of the operators (17.56) and (17.57) is straightforward; we find the states with one or more particles added, provided that the wave function is differentiable. The *ontological* wave functions are not differentiable—they are delta peaks, so particles can only be added as templates, which are to be regarded as probabilistic distributions of ontological states.

Finding the vacuum state, i.e. the quantum ground state itself, is harder. It is that particular superposition of ontological states from which no particles can be removed. Selecting out the annihilation parts of the operators $a^{L,R}(x)$ means that we have to apply the projection operator $\mathcal{P}^-$ on the function $a^L(x)$ and $\mathcal{P}^+$ on $a^R(x)$, where the projection operators $\mathcal{P}^\pm$ are given by

$$\mathcal{P}^\pm a(x) = \sum_{x'} \mathcal{P}^\pm\big(x-x'\big)a\big(x'\big), \tag{17.65}$$

where the functions $P^\pm(y)$ are defined by

$$\mathcal{P}^\pm(y) = \frac{1}{2\pi}\int_0^\pi \mathrm{d}\kappa e^{\pm iy\kappa} = \frac{\pm i}{\pi y} \quad \text{if } y \text{ odd}, \quad \tfrac{1}{2}\delta_{y,0} \quad \text{if } y \text{ even}. \tag{17.66}$$

The state for which the operators $\mathcal{P}^{+}a^{L}(x)$ and $\mathcal{P}^{-}a^{R}(x)$ vanish at all x is the quantum ground state. It is a superposition of all cellular automaton states.

Note that the theory has a Goldstone mode [44, 62], which means that an excitation in which all fields $q(x,t)$, or all automaton variables $Q(x,t)$, get the same constant Q_0 added to them, does not affect the total energy. This is an artefact of this particular model.[1] Note also that the projection operators $\mathcal{P}^{\pm}(x)$ are not well-defined on x- independent fields; for these fields, the vacuum is ill-defined.

17.2 Bosonic Theories in Higher Dimensions?

At first sight, it seems that the model described in previous sections may perhaps be generalized to higher dimensions. In this section, we begin with setting up a scheme that should serve as an approach towards handling bosons in a multiply dimensional space as a cellular automaton. Right-away, we emphasize that a mapping in the same spirit as what was achieved in previous sections and chapters will not be achieved. Nevertheless, we will exhibit here the initial mathematical steps, which start out as deceptively beautiful. Then, we will exhibit, with as much clarity as we can, what, in this case, the obstacles are, and why we call this a failure in the end. As it seems today, what we have here is a loose end, but it could also be the beginning of a theory where, as yet, we were forced to stop half-way.

17.2.1 Instability

We would have been happy with either a discretized automaton or a compactified classical field theory, and for the time being, we keep both options open.

Take the number of space-like dimensions to be d, and suppose that we replace Eqs. (17.38) and (17.39) by

$$Q(\vec{x},t+1)=Q(\vec{x},t)+P\bigl(\vec{x},t+\tfrac{1}{2}\bigr); \tag{17.67}$$

$$P\bigl(\vec{x},t+\tfrac{1}{2}\bigr)=P\bigl(\vec{x},t-\tfrac{1}{2}\bigr)+\sum_{i=1}^{d}\bigl(Q(\vec{x}-\hat{e}_i,t)-2Q(\vec{x},t)+Q(\vec{x}+\hat{e}_i,t)\bigr), \tag{17.68}$$

where $\hat{e}_i$ are unit vectors in the ith direction in space.

Next, consider a given time t. We will need to localize operators in time, and can do this only by choosing the time at which an operator acts such that, at that particular time, the effect of the operator is as concise as is possible. This was why,

[1] Paradoxically, models in two space–time dimensions are known not to allow for Goldstone modes; this theorem [21], however, only applies when there are interactions. Ours is a free particle model.

for the P operators, in Eqs. (17.67) and (17.68), we chose to indicate time as $t \pm \frac{1}{2}$ (where t is integer). Let $\kappa^{\mathrm{op}}(\vec{x}, t+\frac{1}{2})$ be the generator of a shift of $Q(\vec{x}, t)$ and the same shift in $Q(\vec{x}, t+1)$ (so that $P(\vec{x}, t+\frac{1}{2})$ does not shift), while $\xi^{\mathrm{op}}(\vec{x}, t)$ generates identical, negative shifts of $P(\vec{x}, t+\frac{1}{2})$ and of $P(\vec{x}, t-\frac{1}{2})$, without shifting $Q(\vec{x}, t)$ at the same t, and with the signs both as dictated in Eqs. (17.41) and (17.42). Surprisingly, one finds that these operators obey the same equations (17.67) and (17.68): the operation

$$\begin{aligned}
&\kappa^{\mathrm{op}}\big(\vec{x}, t+\tfrac{1}{2}\big) \text{ has the same effect as} \\
&\quad \kappa^{\mathrm{op}}\big(\vec{x}, t+\tfrac{3}{2}\big) - \sum_{i=1}^{2d}\big(\xi^{\mathrm{op}}(\vec{x}+\hat{e}_i, t+1) - \xi^{\mathrm{op}}(\vec{x}, t+1)\big), \\
&\text{and} \quad \xi^{\mathrm{op}}(\vec{x}, t) \text{ has the same effect as} \\
&\quad \xi^{\mathrm{op}}(\vec{x}, t+1) - \kappa^{\mathrm{op}}\big(\vec{x}, t+\tfrac{1}{2}\big)
\end{aligned} \tag{17.69}$$

(where the sum is the same as in Eq. (17.68) but in a more compact notation) and therefore,

$$\xi^{\mathrm{op}}(\vec{x}, t+1) = \xi^{\mathrm{op}}(\vec{x}, t) + \kappa^{\mathrm{op}}\big(\vec{x}, t+\tfrac{1}{2}\big); \tag{17.70}$$

$$\begin{aligned}
\kappa^{\mathrm{op}}\big(\vec{x}, t+\tfrac{1}{2}\big) &= \kappa^{\mathrm{op}}\big(\vec{x}, t-\tfrac{1}{2}\big) \\
&\quad + \sum_{i=1}^{d}\big(\xi^{\mathrm{op}}(\vec{x}-\hat{e}_i, t) - 2\xi^{\mathrm{op}}(\vec{x}, t) + \xi^{\mathrm{op}}(\vec{x}+\hat{e}_i, t)\big),
\end{aligned} \tag{17.71}$$

and, noting that Q and P are integers, while κ^{op} and ξ^{op} are confined to the interval $[0, 2\pi)$, we conclude that again the same equations are obeyed by the real number operators

$$\begin{aligned}
q^{\mathrm{op}}(\vec{x}, t) &\stackrel{?}{=} Q(\vec{x}, t) + \tfrac{1}{2\pi}\xi^{\mathrm{op}}(\vec{x}, t), \quad \text{and} \\
p^{\mathrm{op}}\big(\vec{x}, t+\tfrac{1}{2}\big) &\stackrel{?}{=} 2\pi P\big(\vec{x}, t+\tfrac{1}{2}\big) + \kappa^{\mathrm{op}}\big(\vec{x}, t+\tfrac{1}{2}\big).
\end{aligned} \tag{17.72}$$

These operators, however, do not obey the correct commutation rules. There even appears to be a factor 2 wrong, if we would insert the equations $[Q(\vec{x}), \kappa^{\mathrm{op}}(\vec{x}\,')] \stackrel{?}{=} i\delta_{\vec{x},\vec{x}\,'}$, $[\xi^{\mathrm{op}}(\vec{x}), P(\vec{x}\,')] \stackrel{?}{=} i\delta_{\vec{x},\vec{x}\,'}$. Of course, the reason for this failure is that we have the edge states, and we have not yet restored the correct boundary conditions in ξ, κ space by inserting the phase factors φ, as in Eqs. (17.56), (17.57), or ϕ in (16.14), (16.15). This is where our difficulties begin. These phase factors also aught to obey the correct field equations, and this seems to be impossible to realize.

In fact, there is an other difficulty with the equations of motion, (17.67), (17.68): they are unstable. It is true that, in the continuum limit, these equations generate the usual field equations for smooth functions $q(\vec{x}, t)$ and $p(\vec{x}, t)$, but we now have lattice equations. Fourier transforming the equations in the space variables $\vec{x}$ and time t, one finds

$$\begin{aligned} -2i\big(\sin\tfrac{1}{2}\omega\big)q(\vec{k},\omega) &= p(\vec{k},\omega), \\ -2i\big(\sin\tfrac{1}{2}\omega\big)p(\vec{k},\omega) &= \sum_{i=1}^{d} 2(\cos k_i - 1)q(\vec{k},\omega). \end{aligned} \tag{17.73}$$

This gives the dispersion relation

$$4\sin^2\tfrac{1}{2}\omega = 2(1-\cos\omega) = \sum_{i=1}^{d} 2(1-\cos k_i). \tag{17.74}$$

In one space-like dimension, this just means that $\omega = \pm k$, which would be fine, but if $d > 1$, and k_i take on values close to $\pm\pi$, the r.h.s. of this equation exceeds the limit value 4, the cosine becomes an hyperbolic cosine, and thus we find modes that oscillate out of control, exponentially with time.

To mitigate this problem, we would somehow have to constrain the momenta k_i towards small values only, but, both in a cellular automation where all variables are integers, and in the compactified field model, where we will need to respect the intervals $(-\pi,\pi)$, this is hard to accomplish. Note that we used Fourier transforms on functions such as Q and P in Eqs. (17.67) and (17.68) that take integer values. In itself, that procedure is fine, but it shows the existence of exponentially exploding solutions. These solutions can also be attributed to the non-existence of an energy function that is conserved and bounded from below. Such an energy function does exist in one dimension:

$$\begin{aligned} E = \tfrac{1}{2}\sum_x P^+(x,t)^2 + \tfrac{1}{2}\sum_x \big(Q(x,t) + P^+(x,t)\big) \\ \times \big(2Q(x,t) - Q(x-1,t) - Q(x+1,t)\big), \end{aligned} \tag{17.75}$$

or in momentum space, after rewriting the second term as the difference of two squares,

$$E = \tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}k \Big(\big(\cos\tfrac{1}{2}k\big)^2\big|P^+(k)\big|^2 + 4\big(\sin\tfrac{1}{2}k\big)^2\big|Q(k) + \tfrac{1}{2}P^+(k)\big|^2\Big). \tag{17.76}$$

Up to a factor $\sin k/k$, this is the Hamiltonian (17.25) (since the equations of motion at different k values are independent, conservation of one of these Hamiltonians implies conservation of the other).

In higher dimensions, models of this sort cannot have a non-negative, conserved energy function, and so these will be unstable.

17.2.2 Abstract Formalism for the Multidimensional Harmonic Oscillator

Our PQ procedure for coupled harmonic oscillators can be formalized more succinctly and elegantly. Let us write a time-reversible harmonic model with integer degrees of freedom as follows. In stead of Eqs. (17.67) and (17.68) we write

$$Q_i(t+1) = Q_i(t) + \sum_j T_{ij} P_j\bigl(t+\tfrac{1}{2}\bigr); \tag{17.77}$$

$$P_i\bigl(t+\tfrac{1}{2}\bigr) = P_i\bigl(t-\tfrac{1}{2}\bigr) - \sum_j V_{ij} Q_j(t). \tag{17.78}$$

Here, t is an integer-valued time variable. It is very important that both matrices T and V are real and symmetric:

$$T_{ij} = T_{ji}; \qquad V_{ij} = V_{ji}. \tag{17.79}$$

T_{ij} would often, but not always, be taken to be the Kronecker delta δ_{ij}, and V_{ij} would be the second derivative of a potential function, here being constant coefficients. Since we want Q_i and P_i both to remain integer-valued, the coefficients T_{ij} and V_{ij} will also be taken to be integer-valued. In principle, any integer-valued matrix would do; in practice, we will find severe restrictions.

Henceforth, we shall omit the summation symbol $\sum_j$, as its presence can be taken to be implied by summation convention: every repeated index is summed over.

When we define the translation generators for Q_i and P_i, we find that, in a Heisenberg picture, it is best to use an operator $\kappa_i^{\mathrm{op}}(t+\frac{1}{2})$ to add one unit to $Q_i(t)$ while all other integers $Q_j(t)$ with $j \neq i$ and all $P_j(t+\frac{1}{2})$ are kept fixed. Note that, according to the evolution equation (17.77), this also adds one unit to $Q_i(t+1)$ while all other $Q_j(t+1)$ are kept fixed as well, so that we have symmetry around the time value $t+\frac{1}{2}$. Similarly, we define an operator $\xi_i^{\mathrm{op}}(t)$ that shifts the value of both $P_i(t-\frac{1}{2})$ and $P_i(t+\frac{1}{2})$, while all other Q and P operators at $t-\frac{1}{2}$ and at $t+\frac{1}{2}$ are kept unaffected; all this was also explained in the text between Eqs. (17.68) and (17.69).

So, we define the action by operators κ_i^{op} and ξ_i^{op} by

$$\begin{aligned} e^{-i\kappa_i^{\mathrm{op}}(t+\frac{1}{2})}\bigl|\bigl\{Q_j(t), P_j\bigl(t+\tfrac{1}{2}\bigr)\bigr\}\bigr\rangle &= \bigl|\bigl\{Q'_j(t), P_j\bigl(t+\tfrac{1}{2}\bigr)\bigr\}\bigr\rangle \quad \text{or} \\ e^{-i\kappa_i^{\mathrm{op}}(t+\frac{1}{2})}\bigl|\bigl\{Q_j(t+1), P_j\bigl(t+\tfrac{1}{2}\bigr)\bigr\}\bigr\rangle &= \bigl|\bigl\{Q'_j(t+1), P_j\bigl(t+\tfrac{1}{2}\bigr)\bigr\}\bigr\rangle \quad \text{with} \\ Q'_j(t) = Q_j(t) + \delta_{ji}, \quad & Q'_j(t+1) = q_j(t+1) + \delta_{ji}; \end{aligned} \tag{17.80}$$

$$\begin{aligned} e^{i\xi_i^{\mathrm{op}}(t)}\bigl|\bigl\{Q_j(t), P_j\bigl(t+\tfrac{1}{2}\bigr)\bigr\}\bigr\rangle &= \bigl|\bigl\{Q_j(t), P'_j\bigl(t+\tfrac{1}{2}\bigr)\bigr\}\bigr\rangle \quad \text{or} \\ e^{i\xi_i^{\mathrm{op}}(t)}\bigl|\bigl\{Q_j(t), P_j\bigl(t-\tfrac{1}{2}\bigr)\bigr\}\bigr\rangle &= \bigl|\bigl\{Q_j(t), P'_j\bigl(t-\tfrac{1}{2}\bigr)\bigr\}\bigr\rangle \quad \text{with} \\ & P'_j\bigl(t\pm\tfrac{1}{2}\bigr) = P_j\bigl(t\pm\tfrac{1}{2}\bigr) + \delta_{ji}. \end{aligned} \tag{17.81}$$

The operators $\xi_i^{\mathrm{op}}(t)$ and $\kappa_i^{\mathrm{op}}(t+\frac{1}{2})$ then obey exactly the same equations as $Q_i(t)$ and $P_i(t+\frac{1}{2})$, as given in Eqs. (17.77) and (17.78):

$$\xi_i^{\mathrm{op}}(t+1)=\xi_i^{\mathrm{op}}(t)+T_{ij}\kappa_j^{\mathrm{op}}\bigl(t+\tfrac{1}{2}\bigr); \tag{17.82}$$

$$\kappa_i^{\mathrm{op}}\bigl(t+\tfrac{1}{2}\bigr)=\kappa_i^{\mathrm{op}}\bigl(t-\tfrac{1}{2}\bigr)-V_{ij}\xi_j^{\mathrm{op}}(t). \tag{17.83}$$

The stability question can be investigated by writing down the conserved energy function. After careful inspection, we find that this energy function can be defined at integer times:

$$H_1(t)=\tfrac{1}{2}T_{ij}P_i\bigl(t+\tfrac{1}{2}\bigr)P_j\bigl(t-\tfrac{1}{2}\bigr)+\tfrac{1}{2}V_{ij}Q_i(t)Q_j(t), \tag{17.84}$$

and at half-odd integer times:

$$H_2\bigl(t+\tfrac{1}{2}\bigr)=\tfrac{1}{2}T_{ij}P_i\bigl(t+\tfrac{1}{2}\bigr)P_j\bigl(t+\tfrac{1}{2}\bigr)+\tfrac{1}{2}V_{ij}Q_i(t)Q_j(t+1). \tag{17.85}$$

Note that H_1 contains a pure square of the Q fields but a mixed product of the P fields while H_2 has that the other way around. It is not difficult to check that $H_1(t)=H_2(t+\frac{1}{2})$:

$$H_2-H_1=-\tfrac{1}{2}T_{ij}P_i\bigl(t+\tfrac{1}{2}\bigr)V_{jk}Q_k(t)+\tfrac{1}{2}V_{ij}Q_i(t)T_{jk}P_k\bigl(t+\tfrac{1}{2}\bigr)=0. \tag{17.86}$$

Similarly, we find that the Hamiltonian stays the same at all times. Thus, we have a conserved energy, and that could guarantee stability of the evolution equations.

However, we still need to check whether this energy function is indeed non-negative. This we do by rewriting it as the sum of two squares. In H_1, we write the momentum part (kinetic energy) as

$$\begin{aligned}&\tfrac{1}{2}T_{ij}\bigl(P_i\bigl(t+\tfrac{1}{2}\bigr)+\tfrac{1}{2}V_{ik}Q_k(t)\bigr)\bigl(P_j\bigl(t+\tfrac{1}{2}\bigr)+\tfrac{1}{2}V_{j\ell}Q_\ell(t)\bigr)\\&\quad-\tfrac{1}{8}T_{ij}V_{ik}V_{j\ell}Q_k(t)Q_\ell(t),\end{aligned} \tag{17.87}$$

so that we get at integer times (in short-hand notation)

$$H_1=\vec{Q}\bigl(\tfrac{1}{2}V-\tfrac{1}{8}VTV\bigr)\vec{Q}+\tfrac{1}{2}\bigl(\vec{P}^+ +\tfrac{1}{2}\vec{Q}V\bigr)T\bigl(\vec{P}^+ +\tfrac{1}{2}V\vec{Q}\bigr), \tag{17.88}$$

and at half-odd integer times:

$$H_2=\vec{P}\bigl(\tfrac{1}{2}T-\tfrac{1}{8}TVT\bigr)\vec{P}+\tfrac{1}{2}\bigl(\vec{Q}^- +\tfrac{1}{2}\vec{P}T\bigr)V\bigl(\vec{Q}^- +\tfrac{1}{2}T\vec{P}\bigr), \tag{17.89}$$

where $P^+(t)$ stands for $P(t+\frac{1}{2})$, and $Q^-(t+\frac{1}{2})$ stands for $Q(t)$.

The expression (17.85) for H_2 was the one used in Eq. (17.75) above. It was turned into Eq. (17.89) in the next expression, Eq. (17.76).

Stability now requires that the coefficients for these squares are all non-negative. This has to be checked for the first term in Eq. (17.88) and in (17.89). If V and/or T have one or several vanishing eigenvectors, this does not seem to generate real problems, and we replace these by infinitesimal numbers $\varepsilon>0$. Then, we find that, on the one hand one must demand

$$\langle T\rangle>0,\qquad\langle V\rangle>0; \tag{17.90}$$

while on the other had, by multiplying left and right by V^{-1} and T^{-1}:

$$\langle 4V^{-1} - T\rangle \geq 0, \qquad \langle 4T^{-1} - V\rangle \geq 0. \tag{17.91}$$

Unfortunately, there are not so many integer-valued matrices V and T with these properties. Limiting ourselves momentarily to the case $T_{ij} = \delta_{ij}$, we find that the matrix V_{ij} can have at most a series of 2's on its diagonal and sequences of ± 1 's on both sides of the diagonal. Or, the model displayed above in Eq. (17.67) and (17.68), on a lattice with periodicity N, is the most general multi-oscillator model that can be kept stable by a nonnegative energy function.

If we want more general, less trivial models, we have to search for a more advanced discrete Hamiltonian formalism (see Sect. 19).

If it were not for this stability problem, we could have continued to construct real-valued operators q_i^{op} and p_i^{op} by combining Q_i with ξ_i^{op} and P_i with κ_i^{op}. The operators $e^{iQ\xi_i^{\text{op}}}$ and $e^{-iP\kappa_i^{\text{op}}}$ give us the states $|\{Q_i, P_i\}\rangle$ from the 'zero-state' $|\{0,0\}\rangle$. This means that only one wave function remains to be calculated by some other means, after which all functions can be mapped by using the operators $e^{ia_i p_i}$ and $e^{ib_j q_j}$. But since we cannot obtain stable models in more than 1 space-dimensions, this procedure is as yet of limited value. It so happens, however, that the one-dimensional model is yet going to play a very important role in this work: (super)string theory, see the next section.

17.3 (Super)strings

What follows next, the description of string theory and superstring theory in terms of a cellular automaton, was described by the author in Ref. [126]. In searching for older material, he recently unearthed a letter, written to him by J.G. Russo [69] in March 1993, which contained the details of essentially the same idea concerning the most important case: the bosonic string. Clearly, all priority claims should go to him.

So-far, most of our models represented non-interacting massless particles in a limited number of space dimensions. Readers who are still convinced that quantum mechanical systems will never be explained in terms of classical underlying models, will not be shocked by what they have read until now. After all, one cannot do Gedanken experiments with particles that do not interact, and anyway, massless particles in one spacial dimension do not exhibit any dispersion, so here especially, interference experiments would be difficult to imagine. This next chapter however might make him/her frown a bit: we argue that the bulk region of the (super)string equations can be mapped onto a deterministic, ontological theory. The reason for this can be traced to the fact that string theory, in a flat background, is essentially just a one-space, one-time massless quantum field theory, without interactions, exactly as was described in previous sections.

As yet, however, our (super)strings will not interact, so the string solutions will act as non-interacting particles; for theories with interactions, go to Chaps. 9, 19, and 21.

Superstring theory started off as an apparently rather esoteric and formal approach to the question of unifying the gravitational force with other forces. The starting point was a dynamical system of relativistic string-like objects, subject to the rules of quantum mechanics. As the earliest versions of these models were beset by anomalies, they appeared to violate Lorentz invariance, and also featured excitation modes with negative mass-squared, referred to as "tachyons". These tachyons would have seriously destabilized the vacuum and for that reason had to be disposed of. It turned out however, that by carefully choosing the total number of transverse string modes, or, the dimensions of the space–time occupied by these strings, and then by carefully choosing the value of the intercept $a(0)$, which fixes the mass spectrum of the excitations, and finally by imposing symmetry constraints on the spectrum as well, one could make the tachyons disappear and repair Lorentz invariance [46, 67, 68]. It was then found that, while most excitation modes of the string would describe objects whose rest mass would be close to the Planck scale, a very specific set of excitation modes would be massless or nearly massless. It is these modes that are now identified as the set of fundamental particles of the Standard Model, together with possible extensions of the Standard Model at mass scales that are too large for being detected in today's laboratory experiments, yet small compared to the Planck mass.

A string is a structure that is described by a sheet wiped out in space–time, the string 'world sheet'. The sheet requires two coordinates that describe it, usually called σ and τ. The coordinates occupied in an $n = d + 1$ dimensional space–time, temporarily taken to be flat Minkowski space–time, are described by the symbols[2] $X^\mu(\sigma, \tau)$, $\mu = 0, 1, \dots, d$.

Precise mathematical descriptions of a classical relativistic string and its quantum counterpart are given in several excellent text books [46, 67, 68], and they will not be repeated here, but we give a brief summary. We emphasize, and we shall repeat doing so, that our description of a superstring will not deviate from the standard picture, when discussing the fully quantized theory. We do restrict ourselves to standard *perturbative* string theory, which means that we begin with a simply connected piece of world sheet, while topologically non-trivial configurations occur at higher orders of the string coupling constant g_s. We restrict ourselves to the case $g_s = 0$.

Also, we do have to restrict ourselves to a flat Minkowski background. These may well be important restrictions, but we do have speculations concerning the back reaction of strings on their background; the graviton mode, after all, is as dictated in the standard theory, and these gravitons do represent infinitesimal curvatures in the background. Strings in black hole or (anti)-de Sitter backgrounds are as yet beyond what we can do.

17.3.1 String Basics

An infinitesimal segment $\mathrm{d}\ell$ of the string at fixed time, multiplied by an infinitesimal time segment $\mathrm{d}t$, defines an infinitesimal surface element $\mathrm{d}\Sigma = \mathrm{d}\ell \wedge \mathrm{d}t$. A Lorentz

[2] To stay in line with most literature on string theory, we chose here capital X^μ to denote the (real) space–time coordinates. Later, these will be specified either to be real, or to be integers.

invariant description of $\mathrm{d}\Sigma$ is

$$\mathrm{d}\Sigma^{\mu\nu} = \frac{\partial X^\mu}{\partial\sigma}\frac{\partial X^\nu}{\partial\tau} - \frac{\partial X^\nu}{\partial\sigma}\frac{\partial X^\mu}{\partial\tau}. \tag{17.92}$$

Its absolute value $\mathrm{d}\Sigma$ is then given by

$$\pm\mathrm{d}\Sigma^2 = \tfrac{1}{2}\Sigma^{\mu\nu}\,\mathrm{d}\Sigma^{\mu\nu} = \big(\partial_\sigma X^\mu\big)^2\big(\partial_\tau X^\nu\big)^2 - \big(\partial_\sigma X^\mu\partial_\tau X^\mu\big)^2, \tag{17.93}$$

where the sign distinguishes space-like surfaces $(+)$ from time-like ones $(-)$. The string world sheet is supposed to be time-like.

The string evolution equations are obtained by finding the extremes of the Nambu Goto action,

$$S = -T\int\mathrm{d}\Sigma = -T\int\mathrm{d}\sigma\,\mathrm{d}\tau\,\sqrt{\big(\partial_\sigma X^\mu\partial_\tau X^\mu\big)^2 - \big(\partial_\sigma X^\mu\big)^2\big(\partial_\tau X^\nu\big)^2}, \tag{17.94}$$

where T is the string tension constant; $T = 1/(2\pi\alpha')$.

The light cone gauge is defined to be the coordinate frame (σ,τ) on the string world sheet where the curves $\sigma = \text{const.}$ and the curves $\tau = \text{const.}$ both represent light rays confined to the world sheet. More precisely:

$$\big(\partial_\sigma X^\mu\big)^2 = \big(\partial_\tau X^\mu\big)^2 = 0. \tag{17.95}$$

In this gauge, the Nambu–Goto action takes the simple form

$$S = T\big(\partial_\sigma X^\mu\big)\big(\partial_\tau X^\mu\big) \tag{17.96}$$

(the sign being chosen such that if σ and τ are both pointing in the positive time direction, and our metric is $(-,+,+,+)$), the action is negative. Imposing *both* light cone conditions (17.95) is important to ensure that also the infinitesimal variations of the action (17.96) yield the same equations as the variations of (17.94). They give:

$$\partial_\sigma\partial_\tau X^\mu = 0, \tag{17.97}$$

but we must remember that these solutions must always be subject to the non-linear constraint equations (17.95) as well.

The solutions to these equations are left- and right movers:

$$X^\mu(\sigma,\tau) = X^\mu_L(\sigma) + X^\mu_R(\tau); \qquad \big(\partial_\sigma X^\mu_L\big)^2 = 0, \qquad \big(\partial_\tau X^\mu_R\big)^2 = 0 \tag{17.98}$$

(indeed, one might decide here to rename the coordinates $\sigma = \sigma^+$ and $\tau = \sigma^-$). We now will leave the boundary conditions of the string free, while concentrating on the bulk properties.

The re-parametrization invariance on the world sheet has not yet been removed completely by the gauge conditions (17.95), since we can still transform

$$\sigma \to \tilde{\sigma}(\sigma); \qquad \tau \to \tilde{\tau}(\tau). \tag{17.99}$$

The σ and τ coordinates are usually fixed by using one of the space–time variables; one can choose

$$X^\pm = \big(X^0 \pm X^d\big)/\sqrt{2}, \tag{17.100}$$

to define

$$\sigma = \sigma^+ = X_L^+, \qquad \tau = \sigma^- = X_R^+. \tag{17.101}$$

Substituting this in the gauge condition (17.95), one finds:

$$\partial_\sigma X_L^+ \partial_\sigma X_L^- = \partial_\sigma X_L^- = \tfrac{1}{2} \sum_{i=1}^{d-1} \big(\partial_\sigma X_L^i\big)^2, \tag{17.102}$$

$$\partial_\tau X_R^+ \partial_\tau X_R^- = \partial_\tau X_R^- = \tfrac{1}{2} \sum_{i=1}^{d-1} \big(\partial_\tau X_R^i\big)^2. \tag{17.103}$$

So, the longitudinal variables $X^\pm$, or, X^0 and X^d are both fixed in terms of the $d-1 = n-2$ transverse variables $X^i(\sigma, \tau)$.

The boundary conditions for an open string are then

$$X_L^\mu(\sigma + \ell) = X_L^\mu(\sigma) + u^\mu, \qquad X_R^\mu(\sigma) = X_L^\mu(\sigma), \tag{17.104}$$

while for a closed string,

$$X_L^\mu(\sigma + \ell) = X_L^\mu(\sigma) + u^\mu \qquad X_R^\mu(\tau + \ell) = X_R^\mu(\tau) + u^\mu, \tag{17.105}$$

where ℓ and u^μ are constants. u^μ is the 4-velocity. One often takes ℓ to be fixed, like 2π, but it may be instructive to see how things depend on this free world-sheet coordinate parameter ℓ. One finds that, for an open string, the action over one period is[3]

$$S_{\text{open}} = \tfrac{1}{2} T \int_0^\ell \mathrm{d}\sigma \int_0^\ell \mathrm{d}\tau \, \partial_\sigma X^\mu \partial_\tau X^\mu = \tfrac{1}{2} T u^2. \tag{17.106}$$

For a particle with constant momentum p^μ, the action over an amount of time $X^0 = u^0$ is $S = (\vec{p} \cdot \dot{\vec{q}} - p^0) u^0 = p^\mu u^\mu$, and from that, one derives that the open string's momentum is

$$p^\mu_{\text{open}} = \tfrac{1}{2} T u^\mu. \tag{17.107}$$

For a closed string, the action over one period is

$$S_{\text{closed}} = T \int_0^\ell \mathrm{d}\sigma \int_0^\ell \mathrm{d}\tau \, \partial_\sigma X^\mu \partial_\tau X^\mu = T u^2, \tag{17.108}$$

and we derive that the closed string's momentum is

$$p^\mu_{\text{closed}} = T u^\mu. \tag{17.109}$$

Note that the length ℓ of the period of the two world sheet parameters does not enter in the final expressions. This is because we have invariance under re-parametrization of these world sheet coordinates.

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[3]The factor $1/2$ originates from the fact that, over one period, only half the given domain is covered. Do note, however, that the string's orientation is reversed after one period. One can also understand the factor $1/2$ by regarding the open string as a double strand of a closed string.

Now, in a flat background, the *quantization* is obtained by first looking at the independent variables. These are the transverse components of the fields, being $X^i(\sigma,\tau)$, with $i = 1, 2, \dots, d-1$. This means that these components are promoted to being quantum operators. Everything is exactly as in Sect. 17.1. X^i_L are the left movers, X^μ_R are the right movers. One subsequently postulates that $X^+(\sigma,\tau)$ is given by the gauge fixing equation (17.101), or

$$X^+_L(\sigma) = \sigma, \qquad X^+_R(\tau) = \tau, \tag{17.110}$$

while finally the coordinate $X^-(\sigma,\tau)$ is given by the constraint equations (17.102) and (17.103).

The theory obtained this way is manifestly invariant under rotations among the transverse degrees of freedom, $X^i(\sigma,\tau)$ in coordinate space, forming the space-like group $SO(d-1)$. To see that it is also invariant under other space-like rotations, involving the dth direction, and Lorentz boosts, is less straightforward. To see what happens, one has to work out the complete operator algebra of all fields X^μ, the generators of a Lorentz transformation, and finally their commutation algebra. After a lengthy but straightforward calculation, one obtains the result that the theory is indeed Lorentz invariant provided that certain conditions are met:

- the sequences $J = a(s)$ of string excited modes ("Regge trajectories") must show an intercept $a(0)$ that must be limited to the value $a(0) = 1$ (for open strings), and
- the number of transverse dimensions must be 24 (for a bosonic string) or 8 (for a superstring), so that $d = 25$ or 9, and the total number of space–time dimensions D is 26 or 10.

So, one then ends up with a completely Lorentz invariant theory. It is this theory that we will study, and compare with a deterministic system. As stated at the beginning of this section, many more aspects of this quantized relativistic string theory can be found in the literature.

The operators $X^+(\sigma,\tau)$ and $X^-(\sigma,\tau)$ are needed to prove Lorentz invariance, and, in principle, they play no role in the dynamical properties of the transverse variables $X^i(\sigma,\tau)$. It is the quantum states of the theory of the transverse modes that we plan to compare with classical states in a deterministic theory. At the end, however, we will need X^+ and X^- as well. Of these, X^+ can be regarded as the independent target time variable for the theory, without any further dynamical properties, but then $X^-(\sigma,\tau)$ may well give us trouble. It is not an independent variable, so it does not affect our states, but this variable does control where in space–time our string is. We return to this question in Sect. 17.3.5.

17.3.2 Strings on a Lattice

To relate this theory to a deterministic system [126], one more step is needed: the world sheet must be put on a lattice [58], as we saw in Sect. 17.1.1. How big or how small should we choose the meshes to be? It will be wise to choose these meshes small enough. Later, we will see how small; most importantly, most of our results will turn out to be totally independent of the mesh size $a_{\text{worldsheet}}$. This is because the dispersion properties of the Hamiltonian (17.25) have been deliberately chosen in such a way that the lattice artefacts disappear there: the left- and right movers always go with the local speed of light. Moreover, since we have re-parametrization invariance on the world sheet, in stead of sending $a_{\text{worldsheet}}$ to zero, we could decide to send the length ℓ of the world sheet lattice to infinity. This way, we can keep $a_{\text{worldsheet}} = 1$ throughout the rest of the procedure. Remember that, the quantity ℓ did not enter in our final expressions for the physical properties of the string, not even if they obey boundary conditions ensuring that we talk of open or closed strings. Thus, the physical limit will be the limit $\ell \to \infty$, for open and for closed strings.

We now proceed as in Sects. 17.1.1 and 17.1.2. Assuming that the coordinates x and t used there, are related to σ and τ by[4]

$$\sigma = \tfrac{1}{\sqrt{2}}(x+t), \qquad \tau = \tfrac{1}{\sqrt{2}}(t-x); \qquad x = \tfrac{1}{\sqrt{2}}(\sigma-\tau), \qquad t = \tfrac{1}{\sqrt{2}}(\sigma+\tau), \tag{17.111}$$

we find that the Nambu Goto action (17.96) amounts to $d-1$ copies of the two-dimensional action obtained by integrating the Lagrangian (17.2):

$$\mathcal{L} = \sum_{i-1}^{d-1} \partial_\sigma X^i \partial_\tau X^i, \tag{17.112}$$

provided the string constant T is normalized to one. (Since all $d+1$ modes of the string evolve independently as soon as the on shell constraints (17.101)–(17.103) are obeyed, and we are now only interested in the transverse modes, we may here safely omit the 2 longitudinal modes).

If our units are chosen such that $T = 1$, so that $\alpha' = 1/2\pi$, we can use the lattice rules (17.43) and (17.44), for the transverse modes, or

$$X_L^i(x,t) = p^i(x,t) + X^i(x,t) - X^i(x-1,t); \tag{17.113}$$

$$X_R^i(x,t) = p^i(x,t) + X^i(x,t) - X^i(x+1,t). \tag{17.114}$$

where $P^i(x,t) = \partial_t X^i(x,t)$ (cf Eqs. (17.15) and (17.16), or (17.43) and (17.44)). Since these obey the commutation rules (17.17) and (17.18), or

$$\big[X_L^i, X_R^j\big] = 0; \qquad \big[X_L^i(x), X_L^j(y)\big] = \pm i\delta^{ij} \quad \text{if } y = x \pm 1, \quad \text{else } 0, \tag{17.115}$$

$$\big[X_R^i(x), X_R^j(y)\big] = \mp i\delta^{ij} \quad \text{if } y = x \pm 1, \quad \text{else } 0, \tag{17.116}$$

we can write these operators in terms of integer-valued operators $A_{L,R}^i(x)$ and their associated shift generators $\eta_{L,R}^i$, as in Eqs. (17.56) and (17.57). There, the η basis

[4]With apologies for a somewhat inconsistent treatment of the sign of time variables for the right-movers; we preferred to have τ go in the $+t$ direction while keeping the notation of Sect. 17.1 for left- and right movers on the world sheet.

was used, so that the integer operators $A^i_{L,R}$ are to be written as $-i\partial/\partial\eta^i_{L,R}$. To make an important point, let us, momentarily, replace the coefficients there by α, β, and γ:

$$X^i_L(x) = -i\alpha\frac{\partial}{\partial\eta^i_L(x)} + \beta\left(\frac{\partial}{\partial\eta^i_L(x)}\varphi\bigl(\{\eta^i_L\}\bigr)\right) - \gamma\eta^i_L(x-1), \quad (17.117)$$

and similarly for X^i_R; here $\varphi(\{\eta(x)\})$ is the phase function introduced in Eqs. (17.58) and (17.59).

What fixes the coefficients α, β and γ in these expressions? First, we must have the right commutation relations (17.115) and (17.116). This fixes the product $\alpha\gamma = 1$. Next, we insist that the operators X^i_L are periodic in all variables $\eta^i_L(x)$. This was why the phase function $\varphi(\{\eta\})$ was introduced. It itself is pseudo periodic, see Eq. (17.58). Exact periodicity of X^i_L requires $\beta = 2\pi\gamma$. Finally, and this is very important, we demand that the spectrum of values of the operators $X^-_{L,R}$ runs smoothly from $-\infty$ to ∞ without overlaps or gaps; this fixes the ratios of the coefficients α and γ: we have $\alpha = 2\pi\gamma$. Thus, we retrieve the coefficients:

$$\alpha = \beta = \sqrt{2\pi}; \qquad \gamma = 1/\sqrt{2\pi}. \quad (17.118)$$

The reason why we emphasize the fixed values of these coefficients is that we have to conclude that, in our units, the coordinate functions $X^i_{L,R}(x,t)$ of the cellular automaton are $\sqrt{2\pi}$ times some integers. In our units, $T = 1/(2\pi\alpha') = 1$; $\alpha' = 1/(2\pi)$. In arbitrary length units, one gets that the variables $X^i_{L,R}$ are integer multiples of a *space–time lattice mesh length* $a_{\text{spacetime}}$, with

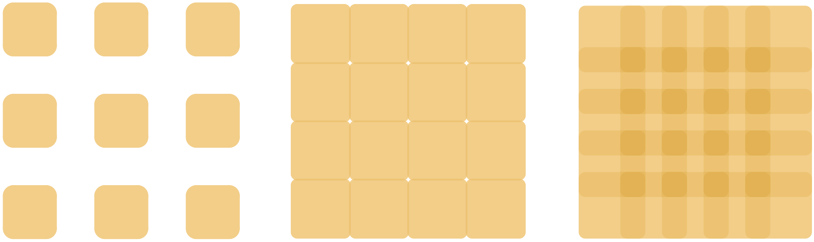
$$a_{\text{spacetime}} = \sqrt{2\pi/T} = 2\pi\sqrt{\alpha'}. \quad (17.119)$$

In Fig. 17.1, the spectrum of the allowed string target space coordinates in the quantum theory is sketched. Only if Eq. (17.119) is exactly obeyed, the classical system exactly matches the quantum theory, otherwise false voids or overlappings appear.[5]

What we find is that our classical string lives on a square lattice with mesh size $a_{\text{spacetime}}$. According to the theory explained in the last few sections of this chapter, the fully quantized bosonic string is entirely equivalent to this classical string; there is a dual mapping between the two. The condition (17.119) on the value of the lattice parameter $a_{\text{spacetime}}$ is essential for this mapping. If string theoreticians can be persuaded to limit themselves to string coordinates that live on this lattice, they will see that the complete set of quantum states of the bosonic string still spans the entire Hilbert space they are used to, while now all basis elements of this Hilbert space propagate classically, according to the discrete analogues of the classical string equations.

Intuitively, in the above, we embraced the lattice theory as the natural ontological system corresponding to a non-interacting string theory in Minkowski space. However, in principle, we could just as well have chosen the compactified theory. This

[5]If the mesh size would be chosen exactly half that of Eq. (17.119), a universal overlap factor of 2^{d-1} would emerge, a situation that can perhaps be accounted for in superstring theory.



a) b) c)

Fig. 17.1 The spectrum of allowed values of the quantum string coordinates X^μ. **a** The case $a_{\text{spacetime}} > 2\pi\sqrt{\alpha'}$, **b** The case $a_{\text{spacetime}} = 2\pi\sqrt{\alpha'}$, **c** The case $a_{\text{spacetime}} < 2\pi\sqrt{\alpha'}$. The squares, representing the ranges of the η parameters, were rounded a bit so as to show the location of possible edge states

theory would assert that the transverse degrees of freedom of the string do not live on $\mathbb{R}^{d-1}$, but on $\mathbb{T}^{d-1}$, a (continuous) torus in $d-1$ dimensions, again with periodicity conditions over lengths $2\pi\sqrt{\alpha'}$, and these degrees of freedom would move about classically.

In Sect. 17.3.5 we elaborate further on the nature of the deterministic string versions.

17.3.3 The Lowest String Excitations

String theory is a quantum field theory on the $1+1$ dimensional world sheet of the string. If this quantum field theory is in its ground state, the corresponding string mode describes the lightest possible particle in this theory. As soon as we put excited states in the world sheet theory, the string goes into excited states, which means that we are describing heavier particles. This way, one describes the mass- or energy spectrum of the string.

In the original versions of the theory, the lightest particle turned out to have a negative mass squared; it would behave as a tachyon, which would be an unwanted feature of a theory.

The more sophisticated, modern versions of the theory are rearranged in such a way that the tachyon mode can be declared to be unphysical, but it still acts as a description of the formal string vacuum. To get the string spectrum, one starts with this unphysical tachyon state and then creates descriptions of the other states by considering the action of creation operators.

To relate these string modes to the ontological states at the deterministic, classical sides of our mapping equation, we again consider the ground state, as it was described in Sect. 17.1.5, to describe the tachyon solution. Thus, the same procedure as in that subsection will have to be applied. Similarly one can get the physical particles by having the various creation operators act on the tachyon (ground) state.

This way, we get our description of the photon (the first spin one state of the open string) and the graviton (the spin 2 excited state of the closed string).

17.3.4 The Superstring

To construct theories containing fermions, it was proposed to plant fermionic degrees of freedom on the string world sheet. Again, anomalies were encountered, unless the bosonic and fermionic degrees of freedom can be united to form super multiplets. Each bosonic coordinate degree of freedom $X^\mu(\sigma,\tau)$ would have to be associated with a fermionic degree of freedom $\psi^\mu(\sigma,\tau)$. This should be done for the left-moving modes independently of the right-moving ones. A further twist can be given to the theory by *only* adding fermionic modes to the left-movers, not to the right movers (or vice versa); this way, chirality can be introduced in string theory, not unlike the chirality that is clearly present in the Standard Model. Such a theory is called a *heterotic string theory*.

Since the world sheet is strictly two-dimensional, we have no problems with spin and helicity within the world sheet, so, here, the quantization of fermionic fields—at least at the level of the world sheet—is simpler than in the case of the 'neutrinos' discussed in Sect. 15.2.3.

Earlier, we used the coordinates σ and τ as light cone coordinates on the world sheet; now, temporarily, we want to use there a space-like coordinate and a time-like one, which we shall call $x^1=x$ and $x^0=t$.

On the world sheet, spinors are 2-dimensional rather than 4-dimensional, and we take them to be Hermitian operators, called Majorana fields, which we write as

$$\psi^{\mu *}(x,t)=\psi^\mu(x,t), \quad \text{and} \quad \psi^{\mu\dagger}(x,t)=\psi^{\mu *T}(x,t) \qquad (17.120)$$

(assuming a real Minkowskian target space; the superscripts ^{*T} mean that if $\psi=\binom{\psi_1}{\psi_2}$ then $\psi^{*T}=(\psi_1^*,\psi_2^*)$).

There are only two Dirac matrices, call them ϱ^0 and ϱ^1, or, after a Wick rotation, ϱ^1 and ϱ^2. They obey

$$\varrho^0=i\varrho^2, \qquad \{\varrho^\alpha,\varrho^\beta\}=\varrho^\alpha\varrho^\beta+\varrho^\beta\varrho^\alpha=2\eta^{\alpha\beta}, \quad (\alpha,\beta=1,2). \qquad (17.121)$$

A useful representation is

$$\varrho^1=\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \qquad \varrho^2=\sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix}; \qquad \varrho^0=\begin{pmatrix}0&1\\-1&0\end{pmatrix}. \quad (17.122)$$

The spinor fields conjugated to the fields $\psi^\mu(x,t)$ are $\overline{\psi}^\mu(x,t)$, here defined by

$$\overline{\psi}^\mu=\psi^{\mu T}\varrho^2, \qquad (17.123)$$

Skipping a few minor steps, concerning gauge fixing, which can be found in the text books about superstrings, we find the fermionic part of the Lagrangian:

$$\mathcal{L}(x,t)=-\sum_{\mu=0}^{d}\overline{\psi}^\mu(x,t)\varrho^\alpha\partial_\alpha\psi^\mu(x,t). \qquad (17.124)$$

Since the ϱ^2 is antisymmetric and fermion fields anti-commute, two different spinors ψ and χ obey

$$\overline{\chi}\psi = \chi^T \varrho^2 \psi = i(-\chi_1\psi_2 + \chi_2\psi_1) = i(\psi_2\chi_1 - \psi_1\chi_2) = \psi^T \varrho^2 \chi = \overline{\psi}\chi. \tag{17.125}$$

Also we have

$$\overline{\chi}\varrho^\mu \psi = -\overline{\psi}\varrho^\mu \chi, \tag{17.126}$$

an antisymmetry that explains why the Lagrangian (17.124) is not a pure derivative. The Dirac equation on the string world sheet is found to be

$$\sum_{\alpha=1}^{2} \varrho^\alpha \partial_\alpha \psi^\mu = 0. \tag{17.127}$$

In the world sheet light cone frame, one writes

$$(\varrho_+ \partial_- + \varrho_- \partial_+)\psi = 0, \tag{17.128}$$

where, in the representation (17.122),

$$\varrho^\pm = \tfrac{1}{\sqrt{2}}\big(\varrho^0 \pm \varrho^1\big), \quad \text{and} \tag{17.129}$$

$$\varrho_+ = -\varrho^- = \sqrt{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \varrho_- = -\varrho^+ = \sqrt{2}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}. \tag{17.130}$$

The solution of Eq. (17.127) is simply

$$\psi^\mu(\sigma,\tau) = \begin{pmatrix} \psi^\mu_L(\sigma) \\ \psi^\mu_R(\tau) \end{pmatrix}. \tag{17.131}$$

Thus, one finds that the fermionic left-movers and right-movers have no further spinor indices.

As is the case for the bosonic coordinate fields $X^\mu_{L,R}(x)$, also the fermionic field components $\psi^\mu_{L,R}$ have two longitudinal modes, $\mu = \pm$, that are determined by constraint equations. These equations are dictated by supersymmetry. So for the fermions also, we only keep the $d-1$ transverse components as independent dynamical fields (d is the number of space-like dimensions in target space).

The second-quantized theory for such fermionic fields has already briefly been discussed in our treatment of the second-quantized 'neutrino' system, in Sect. 15.2.3. Let us repeat here how it goes for these string world sheet fermions. Again, we assume a lattice on the world sheet, while the Dirac equation on the lattice now reduces to a finite-step equation, so chosen as to yield exactly the same solutions (17.131):

$$\begin{aligned}\psi^i(x,t) &= -\tfrac{1}{\sqrt{2}}\varrho^0\big(\varrho^+\psi^i(x-1,t-1) + \varrho^-\psi^i(x+1,t-1)\big),\\ &\quad i = 1,\dots,d-1.\end{aligned} \tag{17.132}$$

The deterministic counterpart is a Boolean variable set $s^i(x,t)$, which we assume to be taking the values ± 1. One may write their evolution equation as

$$s^i(x,t) = s^i(x-1,t-1)s^i(x+1,t-1)s^i(x,t-2). \qquad (17.133)$$

of which the solution can be written as

$$s^i(x,t) = s^i_L(x,t)s^i_R(r,t), \qquad (17.134)$$

where s^i_L and s^i_R obey

$$s^i_L(x,t) = s^i_L(x+1,t-1); \qquad s^i_R(x,t) = s^i_R(x-1,t-1), \qquad (17.135)$$

which is the Boolean analogue of the Dirac equation (17.132).

One can see right away that all basis elements of the Hilbert space for the Dirac equation can be mapped one-to-one onto the states of our Boolean variables. If we would start with these states, there is a straightforward way to construct the anti-commuting field operators $\psi^i_{L,R}(x,t)$ of our fermionic system, the Jordan–Wigner transformation [54], also alluded to in Sect. 15.2.3. At every allowed value of the parameter set (x,i,α), where α stands for L or R, we have an operator $a^i_\alpha(x)$ acting on the Boolean variable $s^i_\alpha(x)$ as follows:

$$a|+\rangle = |-\rangle; \qquad a|-\rangle = 0. \qquad (17.136)$$

These operators, and their Hermitian conjugates $a^\dagger$ obey the mixed commutation - anti-commutation rules

$$\{a^i_\alpha(x), a^i_\alpha(x)\} = 0, \qquad \{a^i_\alpha(x), a^{i\dagger}_\alpha(x)\} = \mathbb{I}, \qquad (17.137)$$

$$[a^i_\alpha(x_1), a^j_\beta(x_2)] = 0, \qquad [a^i_\alpha(x_1), a^{j\dagger}_\beta(x_2)] = 0 \quad \text{if} \begin{cases} & x_1 \neq x_2 \\ \text{and/or} & i \neq j \\ \text{and/or} & \alpha \neq \beta. \end{cases} \qquad (17.138)$$

Turning the commutators in Eq. (17.138) into anti-commutators is easy, if one can put the entire list of variables x, i and α in some order. Call them y and consider the ordering $y_1 < y_2$. Then, the operators $\psi(y)$ can be defined by:

$$\psi(y) \equiv \left(\prod_{y_1<y} s(y_1)\right) a(y). \qquad (17.139)$$

This turns the rules (17.137) and (17.138) into the anti-commutation rules for fermionic fields:

$$\{\psi(y_1), \psi(y_2)\} = 0; \qquad \{\psi(y_1), \psi^\dagger(y_2)\} = \delta(y_1, y_2). \qquad (17.140)$$

In terms of the original variables, the latter rule is written as

$$\{\psi^i_\alpha(x_1), \psi^{j\dagger}_\beta(x_2)\} = \delta(x_1 - x_2)\delta^{ij}\delta_{\alpha\beta}. \qquad (17.141)$$

Let us denote the left-movers L by $\alpha = 1$ or $\beta = 1$, and the right movers R by $\alpha = 2$ or $\beta = 2$. Then, we choose our ordering procedure for the variables $y_1 = (x_1, i, \alpha)$ and $y_2 = (x_2, j, \beta)$ to be defined by

$$\begin{aligned} \text{if } \alpha < \beta \quad & \text{then } y_1 < y_2; \\ \text{if } \alpha = \beta \text{ and } x_1 < x_2 \quad & \text{then } y_1 < y_2; \\ \text{if } \alpha = \beta \text{ and } x_1 = x_2 \text{ and } i < j \quad & \text{then } y_1 < y_2, \\ \text{else } y_1 = y_2 \text{ or } y_1 > y_2. & \end{aligned} \tag{17.142}$$

This ordering is time-independent, since all left movers are arranged before all right movers. Consequently, the solution (17.131) of the 'quantum' Dirac equation holds without modifications in the Hilbert space here introduced.

It is very important to define these orderings of the fermionic fields meticulously, as in Eqs. (17.142). The sign function between brackets in Eq. (17.139), which depends on the ordering, is typical for a Jordan–Wigner transformation. We find it here to be harmless, but this is not always the case. Such sign functions can be an obstruction against more complicated procedures one might wish to perform, such as interactions between several fermions, between right-movers and left-movers, or in attempts to go to higher dimensions (such as in k-branes, where $k > 2$).

At this point, we may safely conclude that our dual mapping between quantized strings and classical lattice strings continues to hold in case of the superstring.

17.3.5 Deterministic Strings and the Longitudinal Modes

The transverse modes of the (non interacting) quantum bosonic and superstrings (in flat Minkowski space–time) could be mapped onto a deterministic theory of strings moving along a target space lattice. How do we add the longitudinal coordinates, and how do we check Lorentz invariance? The correct way to proceed is first to look at the quantum theory, where these questions are answered routinely in terms of quantum operators.

Now we did have to replace the continuum of the world sheet by a lattice, but we claim that this has no physical effect because we can choose this lattice as fine as we please whereas rescaling of the world sheet has no effect on the physics since this is just a coordinate transformation on the world sheet. We do have to take the limit $\ell \to \infty$ but this seems not to be difficult.

Let us first eliminate the effects of this lattice as much as possible. Rewrite Eqs. (17.5) and (17.6) as:

$$p(x,t) = \partial_x k(x,t), \qquad a^{L,R}(x,t) = \partial_x b^{L,R}(x,t); \tag{17.143}$$

$$b^L(x+t) = k(x,t) + q(x,t); \qquad b^R(x-t) = k(x,t) - q(x,t); \tag{17.144}$$

the new fields now obey the equal time commutation rules

$$\big[q(x,t), k\big(x',t\big)\big] = \tfrac{1}{2} i \,\mathrm{sgn}\big(x - x'\big); \tag{17.145}$$

$$\big[b^L(x), b^L(y)\big] = -\big[b^R(x), b^R(y)\big] = -i \,\mathrm{sgn}(x - y), \tag{17.146}$$

where $\mathrm{sgn}(x) = 1$ if $x > 0$, $\mathrm{sgn}(x) = -1$ if $x < 0$ and $\mathrm{sgn}(0) = 0$.

Staying with the continuum for the moment, we cannot distinguish two "adjacent" sites, so there will be no improvement when we try to replace an edge state that is singular at $\eta(x)=\pm\pi$ by one that is singular when this value is reached at two adjacent sites; in the continuum, we expect our fields to be continuous. In any case, we now drop the attempt that gave us the expressions (17.56) and (17.57), but just accept that there is a single edge state at every point. This means that, now, we replace these mapping equations by

$$b^L(x)=\sqrt{2\pi}\,B^L_{\mathrm{op}}(x)+\frac{1}{\sqrt{2\pi}}\zeta^L_{\mathrm{op}}(x), \tag{17.147}$$

$$b^R(x)=\sqrt{2\pi}\,B^R_{\mathrm{op}}(x)+\frac{1}{\sqrt{2\pi}}\zeta^R_{\mathrm{op}}(x), \tag{17.148}$$

where the functions $B^{L,R}$ will actually play the role of the integer parts of the coordinates of the string, and $\zeta^{L,R}_{\mathrm{op}}(x)$ are defined by their action on the integer valued functions $B^{L,R}(x)$, as follows:

$$e^{i\zeta^L(x_1)}\big|\big\{B^{L,R}(x)\big\}\big\rangle=\big|\big\{B'^{L,R}(x)\big\}\big\rangle,\qquad \begin{cases} B'^L(x)=B^L(x)+\theta(x-x_1)\\ B'^R(x)=B^R(x);\end{cases} \tag{17.149}$$

$$e^{i\zeta^R(x_1)}\big|\big\{B^{L,R}(x)\big\}\big\rangle=\big|\big\{B''^{L,R}(x)\big\}\big\rangle,\qquad \begin{cases} B''^L(x)=B^L(x),\\ B''^R(x)=B^R(x)+\theta(x_1-x),\end{cases} \tag{17.150}$$

so that, disregarding the edge state,

$$\big[B^L(x),\zeta^L(y)\big]=-i\theta(x-y),\qquad \big[B^R(x),\zeta^R(y)\big]=-i\theta(y-x). \tag{17.151}$$

This gives the commutation rules (17.146). If we consider again a lattice in x space, where the states are given in the ζ basis, then the operator $B^L_{\mathrm{op}}(x)$ obeying commutation rule (17.151) can be written as

$$B^L_{\mathrm{op}}(x_1)=\sum_{y<x_1}-i\frac{\partial}{\partial\zeta^L(y)}. \tag{17.152}$$

Now the equations of motion of the transverse string states are clear. These just separate into left-movers and right-movers, both for the discrete lattice sites $X^i(\sigma,\tau)$ and for the periodic $\eta^i(\sigma,\tau)$ functions, where $i=1,\dots,d-1$. Also, the longitudinal modes split up into left moving ones and right moving ones. These, however, are fixed by the gauge constraints. In standard string theory, we can use the light cone gauge to postulate that the coordinate variable X^+ is given in an arbitrary way by the world sheet coordinates, and one typically chooses the constraint equations (17.101).

This means that

$$a^+_L(\sigma)=1,\qquad a^+_R(\tau)=1, \tag{17.153}$$

but by simple coordinate transformations $\sigma\to\sigma_1(\sigma)$ and $\tau\to\tau_1(\tau)$, one can choose any other positive function of the coordinate σ (left-mover) or τ (right mover). Now, Eqs. (17.95) here mean that

$$\big(a^\mu_L(\sigma)\big)^2=\big(a^\mu_R(\tau)\big)^2=0, \tag{17.154}$$

so that, as in Eqs. (17.102) and (17.103), we have the constraints

$$a_L^+(\sigma) = \tfrac{1}{2}\sum_{i=1}^{d-1}\big(a_L^i(\sigma)\big)^2, \qquad a_R^+(\tau) = \tfrac{1}{2}\sum_{i=1}^{d-1}\big(a_R^i(t)\big)^2, \tag{17.155}$$

where $a^i_{L,R}(x) = \partial_x b^i_{R,L}(x)$ (see the definitions 17.143).

In view of Eq. (17.152) for the operator $B^i_L(x)$, it is now tempting to write for the longitudinal coordinate $X^+_L:\ \partial_\sigma B^i_L(\sigma) = -i\frac{\partial}{\partial\zeta^i_L(\sigma)}$, so that

$$\partial_\sigma X^+_L(\sigma) \stackrel{?}{=} \tfrac{1}{2}\sum_{i=1}^{d-1}\bigg(-\frac{2\pi\,\partial^2}{\partial\zeta(\sigma)^2} + \frac{1}{2\pi}\big(\partial_\sigma\zeta(\sigma)\big)^2 - 2i\left\{\frac{\partial}{\partial\zeta(\sigma)}, \partial_\sigma\zeta(\sigma)\right\}\bigg), \tag{17.156}$$

but the reader may have noticed that we now disregarded the edge states, which here may cause problems: they occur whenever the functions η^i cross the values $\pm\pi$, where we must postulate periodicity.

We see that we do encounter problems if we want to define the longitudinal coordinates in the compactified classical field theory. Similarly, this is also hard in the discrete automaton model, where we only keep the $B^i_{L,R}$ as our independent ontological variables. How do we take their partial derivatives in σ and τ?

Here, we can bring forward that the gauge conditions (17.153) may have to be replaced by Dirac delta functions, so as to reflect our choice of a world sheet lattice.

These aspects of the string models we have been considering are not well understood. This subsection was added to demonstrate briefly what happens if we study the gauge constraints of the theory to get some understanding of the longitudinal modes, in terms of the ontological states. At first sight it seemed that the compactified deterministic theory would offer better chances to allow us to rigorously derive what these modes look like; it seems as if we can replace the world sheet lattice by a continuum, but the difficulties are not entirely resolved.

If we adopt the cellular automaton based on the integers $B^i_{L,R}$, use of a world sheet lattice is almost inevitable. On the world sheet, the continuum limit has to be taken with much care.

17.3.6 Some Brief Remarks on (Super)string Interactions

As long as our (super)strings do not interact, the effects of the constraints are minor. They tell us what the coordinates $X^-(\sigma,\tau)$ are if we know all other coordinates on the world sheet. In the previous section, our point was that the evolution of these coordinates on the world sheet is deterministic. Our mappings from the deterministic string states onto the quantum string states is one-to-one, apart from the edge states that we choose to ignore. In the text books on string theory, superstring interactions are described by allowing topologically non-trivial world sheets. In practice, this means that strings may exchange arms when they meet at one point, or their end points may join or tear apart. All this is then controlled by a string coupling

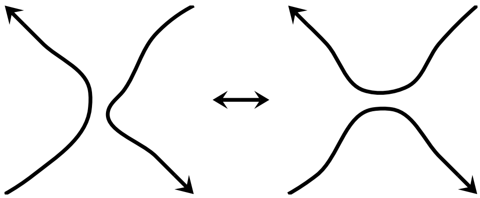


Fig. 17.2 Deterministic string interaction. This interaction takes place whenever two pieces of string meet at one space–time point

constant g_s; an expansion in powers of g_s yields string world sheet diagrams with successively higher topologies.

Curiously, one may very well imagine a string interaction that is deterministic, exactly as the bulk theory obeys deterministic equations. Since, in previous sections, we did not refer to topological boundary conditions, we regard the deterministic description obtained there as a property of the string's 'bulk'.

A natural-looking string interaction would be obtained if we postulate the following:

> Whenever two strings meet at one point on the space–time lattice, they exchange arms, as depicted in Fig. 17.2.

This "law of motion" is deterministic, and unambiguous, provided that both strings are *oriented* strings. The deterministic version of the interaction would not involve any freely adjustable string constant g_s.

If we did not have the problem how exactly to define the longitudinal components of the space–time coordinates, this would complete our description of the deterministic string laws. Now, however, we do have the problem that the longitudinal coordinates are 'quantum'; they are obtained from constraints that are non-linear in the other fields $X^\mu(\sigma, \tau)$, each of which contain integer parts and fractional parts that do not commute.

This problem, unfortunately, is significant. It appears to imply that, in terms of the 'deterministic' variables, we cannot exactly specify where on the world sheet the exchange depicted in Fig. 17.2 takes place. This difficulty has not been resolved, so as yet we cannot produce a 'deterministic' model of interacting 'quantum' strings.

We conclude from our exercise in string theory that strings appear to admit a description in terms of ontological objects, but just not yet quite. The most severe difficulties lie in the longitudinal modes. They are needed to understand how the theory can be made Lorentz invariant. It so happens that local Lorentz invariance is a problem for every theory that attempts to describe the laws of Nature at the Planck scale, so it should not come as a surprise that we have these problems here as well. We suspect that today's incomplete understanding of Lorentz invariance at the Planck scale needs to be repaired, but it may well be that this can only be done in full harmony with the Cellular Automaton Interpretation. What this section suggests us is that this cannot be done solely within the framework of string theory, although strings may perhaps be helpful to lead us to further ideas.

An example of a corner of string theory that has to be swept clean is the black hole issue. Here also, strings seem to capture the physical properties of black holes partly but not completely; as long as this is the case one should not expect us to be able to formulate a concise ontological theory. This is why most parts of this book concentrate on the general philosophy of the CAI rather than attempting to construct a complete model.